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Microwave Hall Effect in a TE_{11p} Cylindrical Cavity

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***Abstract*—A microwave Hall effect signal produced by a TE_{11p} degenerate cylindrical cavity is considered. The analysis presented identifies the cavity end walls as the source of the Hall signal, and a formula is derived relating the Hall output power to the dimensions of the cavity. Experimental results from the measurement of the empty cavity signal at 9.5 GHz are reported.**

I. Introduction

The microwave Hall effect method has recently been introduced in measurements of the mobility values of metals [1]. The method utilizes a resonant cavity with doubly degenerate modes. When a static magnetic field is applied in the direction of the symmetry axis, the electrical conductivity of the metal ceases to be scalar and becomes tensorial. The tensorial conductivity connects the otherwise mutually independent modes of the cavity and results in power transfer from the input port to the output port. This power transfer is usually referred to as the empty-cavity signal. Theoretical analysis has shown that an empty-cavity signal produced by a rectangular degenerate cavity operating at its fundamental modes is given as a function of cavity dimensions [1].

The microwave Hall effect method has been used extensively in the measurement of low Hall mobility values in a variety of materials [2]–[4]. The test sample is usually held at the cavity E-field antinode, and the Hall effect manifests itself as a rotation in the plane of polarization of the microwaves as they pass through the sample. However, the empty-cavity signal may obscure measurement of the Hall effect in low-mobility materials. This background signal is superimposed on the signal from the sample and may become dominant for materials of low mobility. The empty cavity signal must therefore be minimized in order to accurately measure the Hall effect in such materials.

In this paper, the Hall effect in the cavity end walls is described and a quantitative description of the empty cavity signal is given. The calculation is based on Slater's treatment [5] of the expanded electromagnetic fields in terms of the two orthogonal TE_{11p} modes in a cylindrical cavity. An expression for the empty-cavity signal is then formulated in terms of cavity dimensions. Experimental results from measurements at 9.5 GHz have been obtained.

II. Theory

The cavity treated in this paper is a cylindrical one in which only doubly degenerate TE_{11p} modes are excited. The cavity, of length d and radius a, is iris-coupled to matched waveguide transmission lines such that in principle there is complete isolation between input and output waveguides. The coupling slots S_1 and S_2 connect the cavity with the mutually orthogonal waveguides as shown in Fig. 1. In the absence of the magnetic field, the power fed from the oscillator to the cavity via waveguide 1 does not arrive at the receiver coupled to waveguide 2. If an external magnetic field, B, is applied along the z axis, then in addition to the dominant mode (I) excited in the cavity an additional mode (II) will be excited as a consequence of the Hall effect in the cavity end walls. A portion of the incident power, P_1, begins to arrive at the receiver; this is referred to as the empty-cavity Hall output power, P_2. The electromagnetic field in a cylindrical cavity operating in the TE_{11p} mode may be written as

$$E_r = (-E_0/k_c r) J_1(k_c r) \sin(p\pi z/d) \begin{cases} \sin\phi \\ \cos\phi \end{cases} \tag{1a}$$

$$E_\phi = E_0 J_1'(k_c r) \sin(p\pi z/d) \begin{cases} \cos\phi \\ \sin\phi \end{cases} \tag{1b}$$

$$H_r = (jp\pi E_0/\omega\mu_0 d) J_1'(k_c r) \cos(p\pi z/d) \begin{cases} \cos\phi \\ \sin\phi \end{cases} \tag{1c}$$

$$H_\phi = (jp\pi E_0/\omega\mu_0 k_c r) J_1(k_c r) \cos(p\pi z/d) \begin{cases} -\sin\phi \\ \cos\phi \end{cases} \tag{1d}$$

$$H_z = (jk_c E_0/\omega\mu_0) J_1(k_c r) \sin(p\pi z/d) \begin{cases} \cos\phi \\ \sin\phi \end{cases} \tag{1e}$$

where E_0 is the maximum electric field in the cavity, k_c the cutoff wavenumber, μ_0 the permeability of free space, and ω the angular frequency. $J_1(k_c r)$ is a cylindrical Bessel function of the first order and $J_1'(k_c r)$ is its first derivative. The total electric field, $\boldsymbol{E}$, in the cavity may be expressed in the form of

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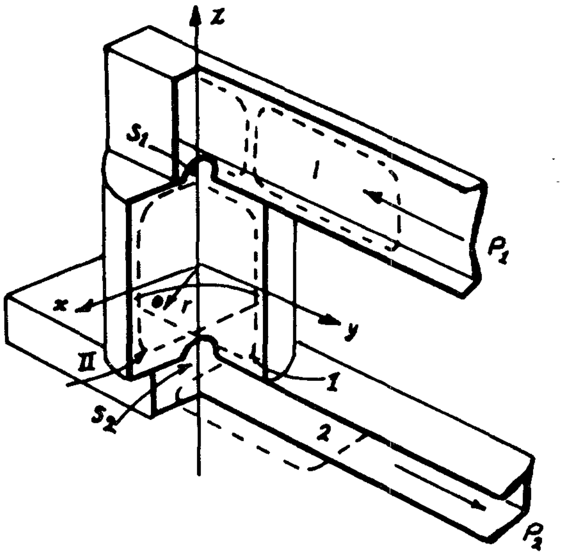

Fig. 1. Cylindrical bimodal cavity with input and output waveguides.

the superposition of the natural modes:

$$E = E_0[(J_1(k_c r)/k_c r)\sin\phi \boldsymbol{r} + J_1'(k_c r)\cos\phi \boldsymbol{\phi}] \cdot \sin(p\pi z/d)e_{\rm I} + E_0[(-J_1(k_c r)/k_c r)\cos\phi \boldsymbol{r} + J_1'(k_c r)\sin\phi\boldsymbol{\phi}] \cdot\sin(p\pi z/d)e_{\rm II} \quad (2)$$

where $e_{\rm I}$ and $e_{\rm II}$ are their amplitudes, respectively, representing the expansion between the voltage in the coupling waveguides and the two modes. The field in the cavity may also be expanded in terms of the doubly degenerate TE_{11p} modes such that [1]

$$\{j[(\omega/\omega_i)-(\omega_i/\omega)]+(1+j)/Q_i\}e_i + (1/2\omega_i U_i)\int \boldsymbol{J}\cdot \boldsymbol{E}_i dv = (i_n \nu_n/2\omega_i U_i) \quad (3)$$

where $i = {\rm I, II}$ denotes the mode of interest, n denotes the cavity port number, Q is the unloaded quality factor, i_n the current in the waveguide, ν_n a voltage expansion coefficient of the waveguide, and U_i the stored energy in the cavity. The current density inside the cavity end walls is the main term of interest in (3). This current is induced at the surface of the conductor when an electromagnetic wave is incident on a metal of finite conductivity σ_0. In the presence of the static magnetic field, B, the conductivity of the cavity end walls ceases to be scalar and becomes tensorial. The total current density is given by

$$\boldsymbol{J} = [j(1+j)\pi p E_0/\omega\mu_0 d]e^{-(1+j)z/\delta}\{[(\sigma_0 J_1(k_c r)/k_c r)(-\sin\phi e_{\rm I}+\cos\phi e_{\rm II}) + (\sigma_1 J_1'(k_c r)(\cos\phi e_{\rm I} - \sin\phi e_{\rm II})]\boldsymbol{r} + [(\sigma_1 J_1(k_c r)/k_c r)(\sin\phi e_{\rm I} - \cos\phi e_{\rm II}) + (\sigma_0 J_1'(k_c r))\cdot(-\cos\phi e_{\rm I} - \sin\phi e_{\rm I})]\boldsymbol{\phi}\} \quad (4)$$

where δ is the skin depth of the cavity end wall material. For a metal, in the case where $\mu_H B \ll 1$, $\sigma_1 = \sigma_0\mu_H B$, where μ_H is the Hall mobility of free electrons in the material of the cavity end walls. If the cavity is excited at port $n = 1$, then combining (2), (3), and (4) gives

$$\{j[(\omega/\omega_{\rm I})-(\omega_{\rm I}/\omega)]+(1+j)/Q_{\rm I}\}e_{\rm I} - \{2(1+j)\pi^2 p^2\mu_H B\delta/\omega^2\mu_0\epsilon_0 d^3(k_c^2a^2-1)\}e_{\rm II} = (i_1\nu_1/2\omega_{\rm I}U_{\rm I}) \quad (5a)$$

and similarly for $n = 2$:

$$\{j[(\omega/\omega_{\rm II})-(\omega_{\rm II}/\omega)]+(1+j)/Q_{\rm II}\}e_{\rm II} - \{2(1+j)\pi^2 p^2\mu_H B\delta/\omega^2\mu_0\epsilon_0 d^3(k_c^2a^2-1)\}e_{\rm I} = (i_2\nu_2/2\omega_{\rm II}U_{\rm II}) \quad (5b)$$

where ϵ_0 is the permittivity of free space. Rearranging (5) and using the proper definitions of the various terms [1], [5], the S matrix can be obtained, the transmission coefficient of which is related to the Hall output power by [1]

$$(P_2/P_1)^{1/2} = (\mu_H B/R)[(1-\Gamma_{01})(1+\Gamma_{01})(1-\Gamma_{02})(1+\Gamma_{02})]^{1/2} \quad (6)$$

where Γ_{01} and Γ_{02} are reflection coefficients at the cavity input and output ports and R is a cavity geometry coefficient given by

$$R = (1/2)[2(p_{11}'^2-1)+(d/a)+(p_{11}'^4/p^2\pi^2)(d/a)^3] \quad (7)$$

where $p_{11}' = k_c a = 1.841$ is the first root of $J_1'(k_c a) = 0$. Similar expressions were derived for a square-section rectangular bimodal cavity operating at its fundamental TE_{011} and TE_{101} modes [1].

III. Discussion

An essential requirement for microwave Hall effect measurement in low-mobility materials is a high-Q-factor cavity, which produces a low empty cavity signal. This is particularly useful for biological and polymeric materials with a typical value of Hall mobility of the order 0.1 $cm^2/V\cdot s$. The maximum sensitivity of detecting a Hall effect signal produced by a low-mobility material is obtained when the sample is placed symmetrically, with respect to the two modes, inside a critically coupled cavity. A sample Hall effect signal of the order of -112 dB must be detected at 1 T in order to attribute it to the material under test and not to the cavity [3]. This requires the use of a long, thin cavity with a cavity geometry coefficient value, R, of 1200, i.e., an (a/d) ratio of 0.1, in order to produce a signal of this magnitude. This corresponds to a cavity length of 89 mm for a 10 GHz resonant frequency, a requirement that may be restricted by the separation of the pole pieces of the magnet employed in the measurement system. It is also difficult to discern such a low signal from stray couplings of the detection system, which may suffer from repeatability and stability problems.

For a brass X-band cylindrical TE_{111} cavity of length 21 mm and radius 14 mm, Sayed and Westgate [2] have measured the minimum detectable mobility to be 0.07 $cm^2/V\cdot s$. They have also reported measurement of the Hall mobility values of several crystals of the order 0.1 $cm^2/V\cdot s$ [2] without reference to the cavity end-wall contribution. It is estimated that such contribution could have been no lower than -70 dB at 1 T for a critically coupled cavity. This corresponds in theory to a minimum detectable sample mobility of 12 $cm^2/V\cdot s$. However, their cavity was always undercoupled [2], which resulted in a reduction of the measurement sensitivity and may imply that an appreciable empty-cavity signal was not observed.

Cross and Pethig [3] have assigned the lower limit of detection to 0.3 $cm^2/V \cdot s$ for a copper Q-band cylindrical TE_{113} cavity of length 13.8 mm and radius 13.9 mm which produced an effective n-type empty-cavity signal of -113 dB at 1 T. However, the reflection coefficient values at both ports were not given [3]. For critical coupling, such an empty cavity could have produced a signal as large as -65 dB at 1 T. It may therefore be concluded that the reported value of -113 dB may have been obtained with a very lightly undercoupled cavity.

The results reported by Eley and Lockhart [4] agree closely with theory, even though no values of reflection coefficient were given. Several TE_{111} cylindrical cavities, all about 20 mm long and 15 mm in radius and resonating between 9.2 and 9.5 GHz with Q's of 9500, were employed in their work [4]. The cavities were made of copper and brass and gave n-type empty-cavity signals of -82 to -84 dB at 1.21 T. The theory predicts that such cavities may produce a maximum of -68 dB at 1.21 T for a critically coupled cavity.

An X-band cylindrical TE_{111} cavity similar to the one used by Sayed and Westgate [2] which employed coaxial probes in its coupling has been designed. The dimensions of the cavity were obtained with the aid of mode charts and mode shape factors [6]. A radius-to-length ratio (a/d) of 0.675 was chosen since it gives a maximum Q factor. For a resonant frequency of 9.5 GHz, the radius and length of the cavity were determined to be 14 mm and 21 mm respectively. The separation between the TE_{111} modes and the interfering modes, TM_{010} and TM_{011}, were 1.29 and 1.38 GHz respectively. At critical coupling this cavity produces an empty-cavity signal of -71 dB at 1 T, which corresponds in theory to a minimum detectable sample Hall mobility of 11 $cm^2/V \cdot s$. It is concluded that smaller mobility values than this cannot be reliably measured using this cavity.

IV. Conclusion

The analysis presented here identifies the end walls of the cylindrical degenerate cavity as the main source of the empty-cavity Hall signal. This signal, which is given as a function of cavity dimensions, limits the application of the microwave Hall measurement method to low-mobility materials, as confirmed by measurements carried out at 9.5 GHz using a TE_{111} cavity. The microwave Hall effect could be an ideal method for investigating the Hall mobility of high-conductivity materials, particularly metals, superconductors, and low-dimensional structures.

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